DEPARTMENT OF
ELECTRICAL &
COMPUTER ENGINEERING

INSTITUTE FOR
SYSTEMS RESEARCH
A. JAMES CLARK SCHOOL OF ENGINEERING

Payoff dynamics and higher-order learning in population games: Stability Analysis: Potential and Contractive Games

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Tutorial co-organized with S. Park (Princeton U) and J. S. Shamma (KAUST)

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Outline

Recap of Basic Formulation

Infinite Population Limit Approximation

Stochastic Model and Convergence Guarantees

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Nash Stationarity and Positive Correlation

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Characterizing GAS of Nash Equilibria Set

Population Games: Basic Formulation

Without loss of generality we consider a single population

Strategy set for the population is $\{1, \ldots, n\}$

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Population state for a population with N *agents*

$$X^N(t) = \begin{bmatrix} X_1^N(t) \\ \vdots \\ X_n^N(t) \end{bmatrix}$$

$X_i^N(t)$ *is the portion of the population selecting strategy* i *at time* t.

We assume unit mass $\sum_{i=1}^n X_i^N(t) = 1$

right-continuous jump process

strategy revisions occur at the jump times

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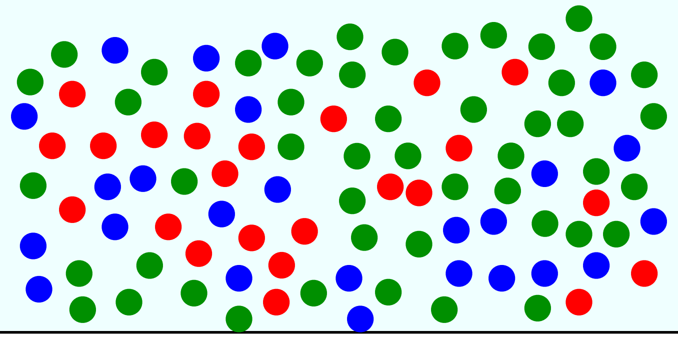
Example for $n = 3$ *and* $N = 100$

$$X^{100}(t^*) = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix}$$

- *strategy 1*
- *strategy 2*
- *strategy 3*

population state at time t^*



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Rules

Each agent selects a strategy at a time

$$\varrho P(\text{agent switches from } i \text{ to } j \mid \text{agent can switch at time } t^*)$$
$$= \mathcal{T}_{ij}(X^N(t), P^N(t))$$

probabilistic model of protocol

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strategy profile set $\quad \mathbb{X} \stackrel{\text{def}}{=} \left\{ z \in \mathbb{R}_+^n \;\middle|\; \sum_{j=1}^n z_j = 1 \right\}$

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The payoff of a population game is specified as follows, where $\mathcal{F} : \mathbb{X} \to \mathbb{R}^n$ is a continuously differentiable map.

$$P^N(t) := \mathcal{F}\big(X^N(t)\big), \qquad P^N(t) = \begin{bmatrix} P_1^N(t) \\ \vdots \\ P_n^N(t) \end{bmatrix}$$

$P_i^N(t)$ is the payoff of strategy i at time t

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Notation:

We use z in $\mathbb{X}$ to represent a given possible population state

We use r in $\mathbb{R}^n$ to represent a given possible payoff vector

T … $\mathbb{R}^n$ is a continuously differentiable map.

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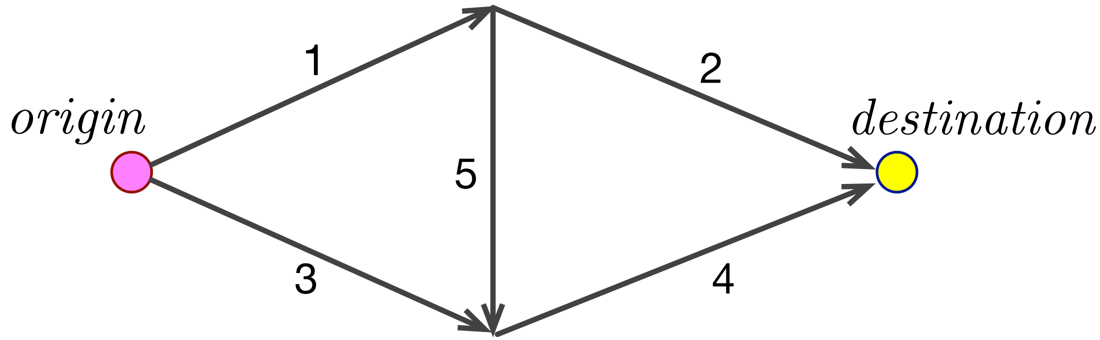
$P_i^N(t)$ *is the payoff of strategy* i *at time* t

The Nash equilibria set for $\mathcal{F}$ *is defines as follows:*

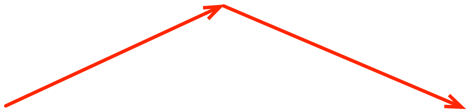
$$\mathbb{NE}(\mathcal{F}) := \left\{ z \in \mathbb{X} \;\middle|\; z^T \mathcal{F}(z) \geq \bar{z}^T \mathcal{F}(z),\ \bar{z} \in \mathbb{X} \right\}$$

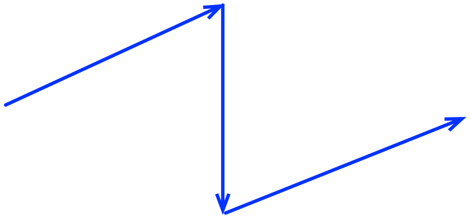
Population Games: Basic Formulation

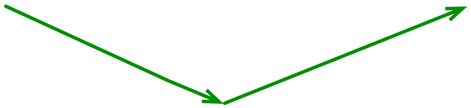
Example: 3-strategy congestion game



$\mathcal{D}_i : \mathbb{X} \rightarrow \mathbb{R}$ *is delay as a* C^1 *increasing function of utilization in link* i

strategy 1  $\mathcal{F}_1(z) = -\mathcal{D}_1(z_1 + z_2) - \mathcal{D}_2(z_1)$

strategy 2  $\mathcal{F}_2(z) = -\mathcal{D}_1(z_1 + z_2) - \mathcal{D}_4(z_2 + z_3) - \mathcal{D}_5(z_2)$

strategy 3  $\mathcal{F}_3(z) = -\mathcal{D}_3(z_3) - \mathcal{D}_4(z_2 + z_3)$

Population Games: Basic Formulation

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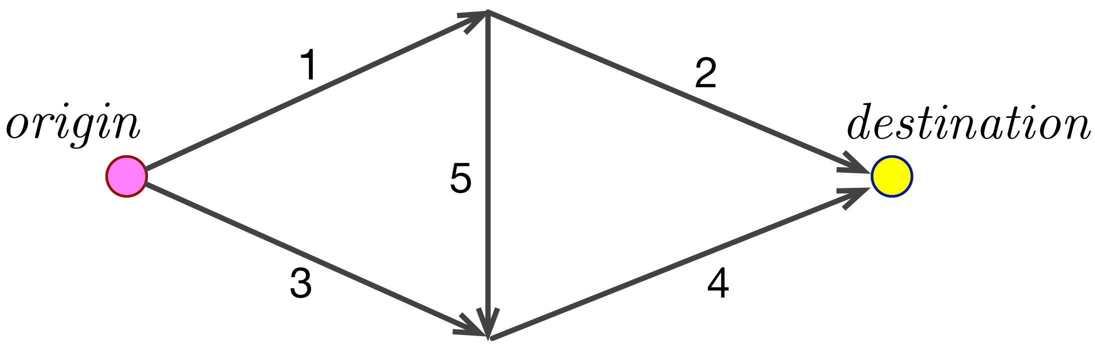

$$\mathcal{F}_1(z) = -\mathcal{D}_1(z_1 + z_2) - \mathcal{D}_2(z_1)$$
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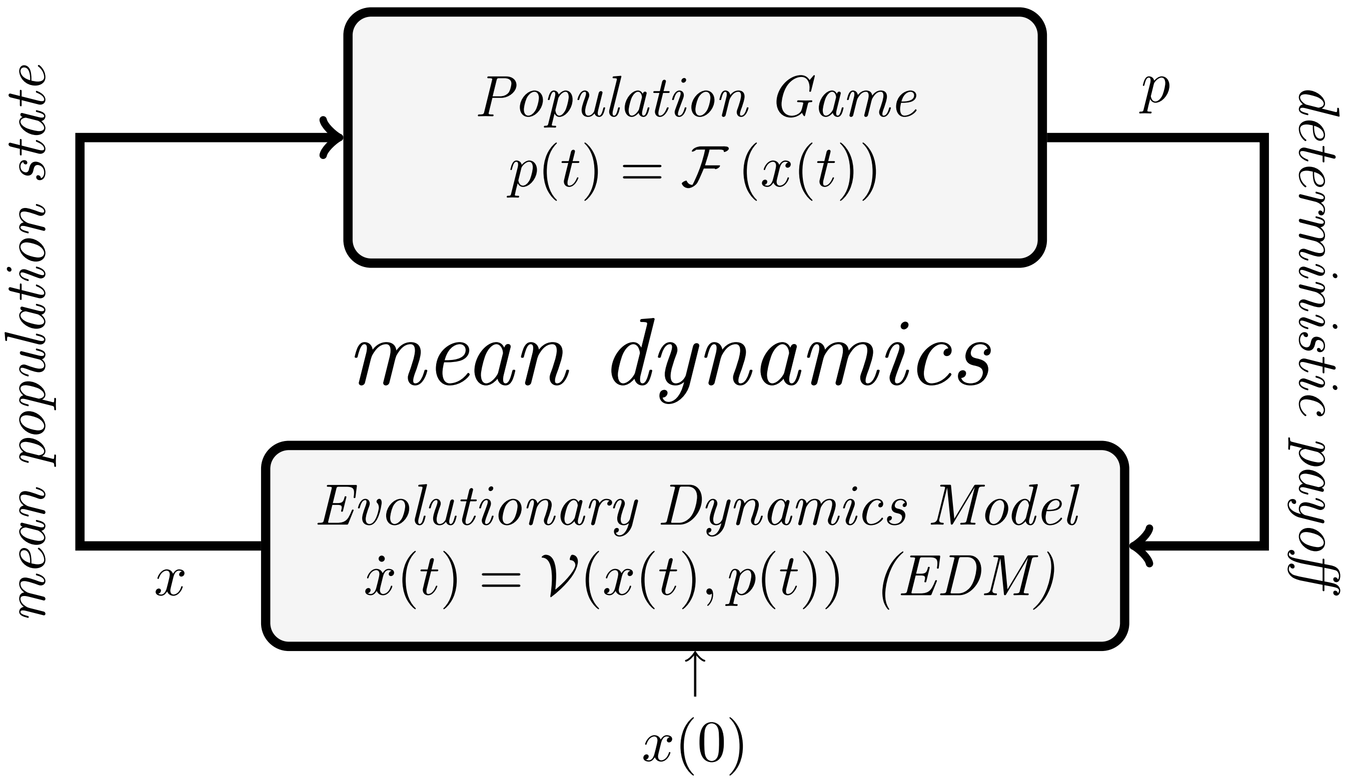
$$\mathcal{F}_3(z) = -\mathcal{D}_3(z_3) - \mathcal{D}_4(z_2 + z_3)$$

$\mathcal{F}$ is a potential strictly concave game

$$D\mathcal{F}(z) = \begin{bmatrix} -\dot{\mathcal{D}}_1(z_1 + z_2) - \dot{\mathcal{D}}_2(z_1) & -\dot{\mathcal{D}}_1(z_1 + z_2) & 0 \\ -\dot{\mathcal{D}}_1(z_1 + z_2) & -\dot{\mathcal{D}}_1(z_1 + z_2) - \dot{\mathcal{D}}_4(z_2 + z_3) - \dot{\mathcal{D}}_5(z_2) & -\dot{\mathcal{D}}_4(z_2 + z_3) \\ 0 & -\dot{\mathcal{D}}_4(z_2 + z_3) & -\dot{\mathcal{D}}_3(z_3) - \dot{\mathcal{D}}_4(z_2 + z_3) \end{bmatrix}$$

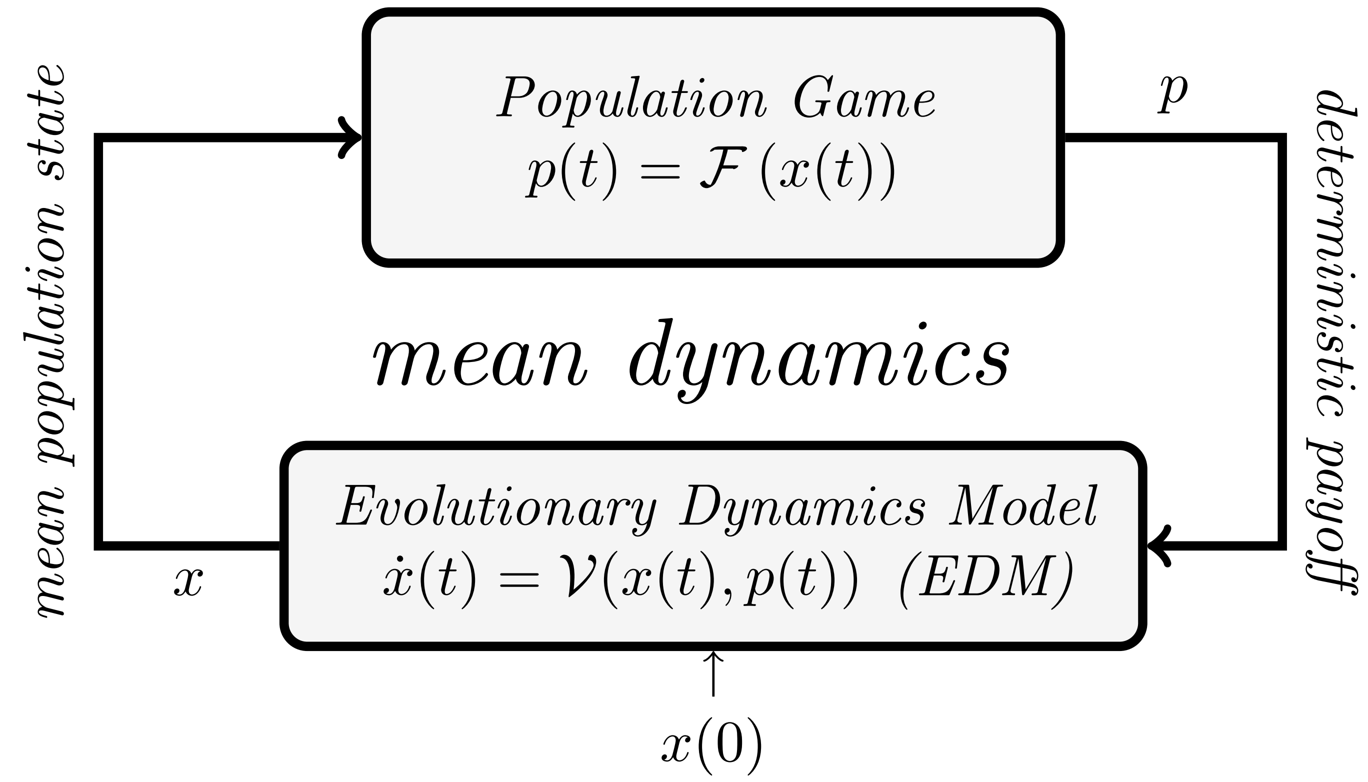
Population Games: Mean Dynamics

Deterministic approximation for large population limit $(N \to \infty)$



Population Games: Mean Dynamics

Deterministic approximation for large population limit ($N \to \infty$)



The EDM $\mathcal{V} : \mathbb{X} \times \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous and guarantees that every solution of the mean dynamics remains in $\mathbb{X}$.

Population Games: Mean Dynamics

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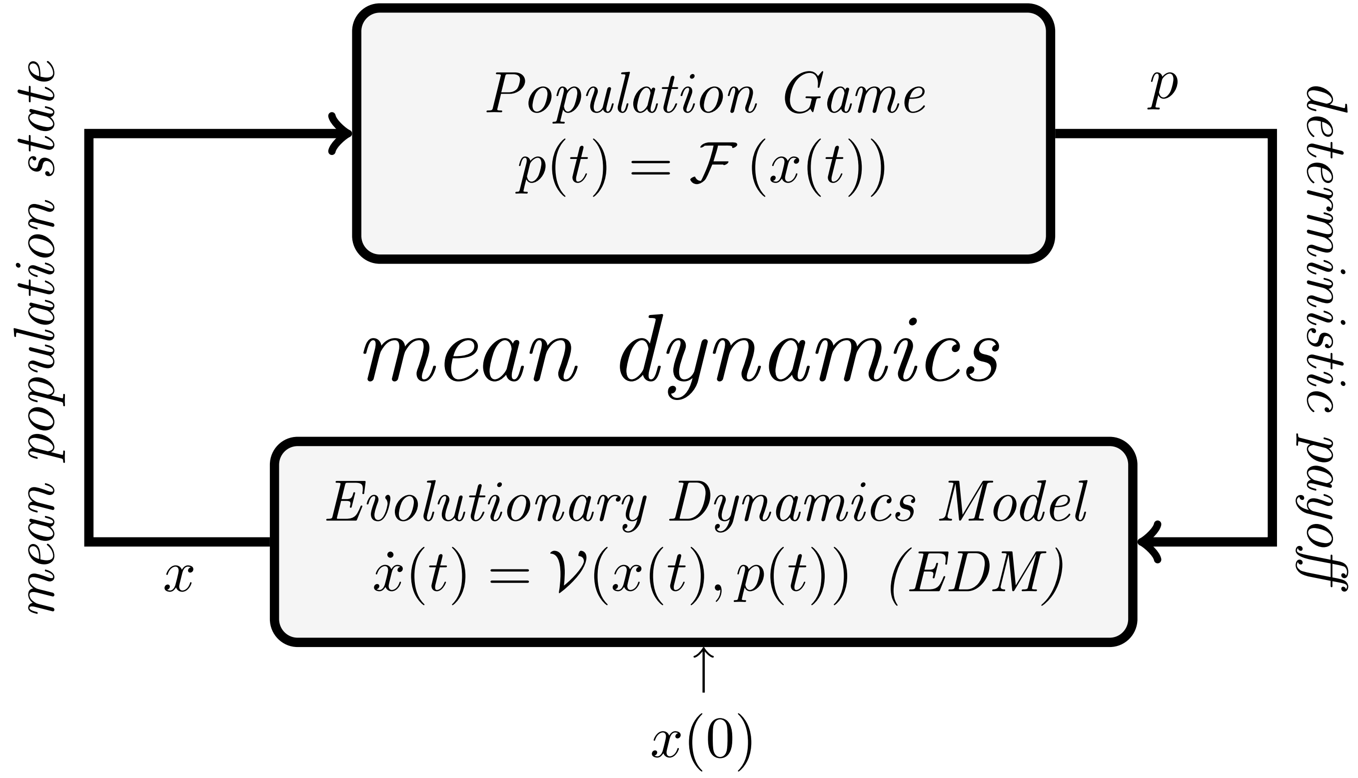

$$\mathcal{V}_i(z,r) := \sum_{j=1}^{n} z_j \mathcal{T}_{ji}(z,r) - \left(\sum_{j=1}^{n} \mathcal{T}_{ij}(z,r) \right) z_i,$$

$$z \in \mathbb{X},\ r \in \mathbb{R}^n,\ 1 \le i \le n$$

probabilistic model of protocol

(EDM) Evolutionary dynamics model

The Stability of a Dynamic Model of Traffic Assignment—An Application of a Method of Lyapunov

MICHAEL J. SMITH

Department of Mathematics, University of York, Heslington, York, England

This paper considers a dynamic model of traffic assignment in which drivers change their route choices to take advantage of cheaper routes. Using a method due to Lyapunov, we show that if the cost-flow function is monotone and there are no explicit capacity restrictions then any solution trajectory of our dynamical system converges to the set of Wardrop equilibria as time passes.

INTRODUCTION

THE paper considers a dynamical model of route-choice. We apply a method of Lyapunov to show that the set of Wardrop equilibria is nonempty and that our dynamical model converges to the set of Wardrop equilibria as time passes, whatever starting flow is chosen, provided the cost-flow function is monotone and smooth.

The dynamical model considered here is an appropriate starting point for a more wide-ranging study of stability in traffic assignment.

It is clear that stability questions are important in real life, and hence that the traffic analyst should be concerned with dynamics *even when demand does not vary with time.* For instance, unstable traffic equilibria are unlikely to persist in practice and so the analyst should check on the stability of his theoretical solutions to an assignment problem. Or, again, the possibility of many solutions to an equilibrium problem forces the analyst to consider the dynamics resulting from different starting, or initial, assignments so as to determine whether there are multiple equilibria and, if so, to determine those equilibria which are likely to arise in practice. As a final example, dynamical considerations are also essential if it is thought that there is more than one equilibrium, that one equilibrium is better (in some sense) than others, and that control

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0041-1655/84/1803-0245 $01.25

$$\mathcal{V}_i(z,r) := \sum_{j=1}^{n} z_j \mathcal{T}_{ji}(z,r) - \left(\sum_{j=1}^{n} \mathcal{T}_{ij}(z,r)\right) z_i,$$

$$z \in \mathbb{X},\ r \in \mathbb{R}^n,\ 1 \le i \le n$$

(Smith EDM) *The Smith EDM is specified by the following Smith IPC protocol:*

$$\mathcal{T}_{ij}^{Smith}(z,r) := [r_j - r_i]_+, \quad r \in \mathbb{R}^n, \quad z \in \mathbb{X}$$

$$\mathcal{V}_i^{Smith}(z,r) := \sum_{j=1}^{n} z_j [r_i - r_j]_+ - \left(\sum_{j=1}^{n} [r_j - r_i]_+\right) z_i$$

A suitable stochastic model

- $\mathcal{T} : \mathbb{X} \times \mathbb{R}^n \to \mathbb{R}_+^{n \times n}$ is Lipschitz continuous.
- Revision clocks are independent and Poisson with rate ϱ.
- Randomizations are independent across agents.

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$$\begin{bmatrix} X^N(t) \\ P^N(t) \end{bmatrix} \text{ is a right-continuous Markov jump process.}$$

$$\Pr\left(X_j^N(\tau) = X_j^N(\tau^-) + \tfrac{1}{N} \ \&\ X_i^N(\tau) = X_i^N(\tau^-) - \tfrac{1}{N} \;\middle|\; X^N(\tau^-) = z, \text{ transition at time } \tau \right) = z_i \frac{\mathcal{T}_{ij}\Big(z, P^N(\tau^-)\Big)}{\varrho}$$

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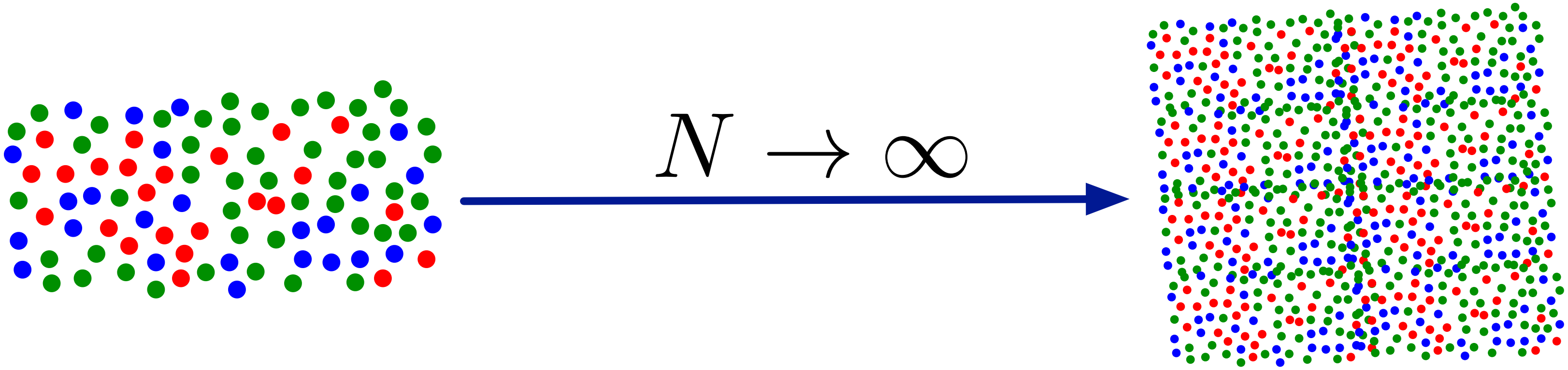

Taking The Limit As $N \to \infty$

- $\mathcal{T} : \mathbb{X} \times \mathbb{R}^n \to \mathbb{R}_+^{n \times n}$ is Lipschitz continuous.
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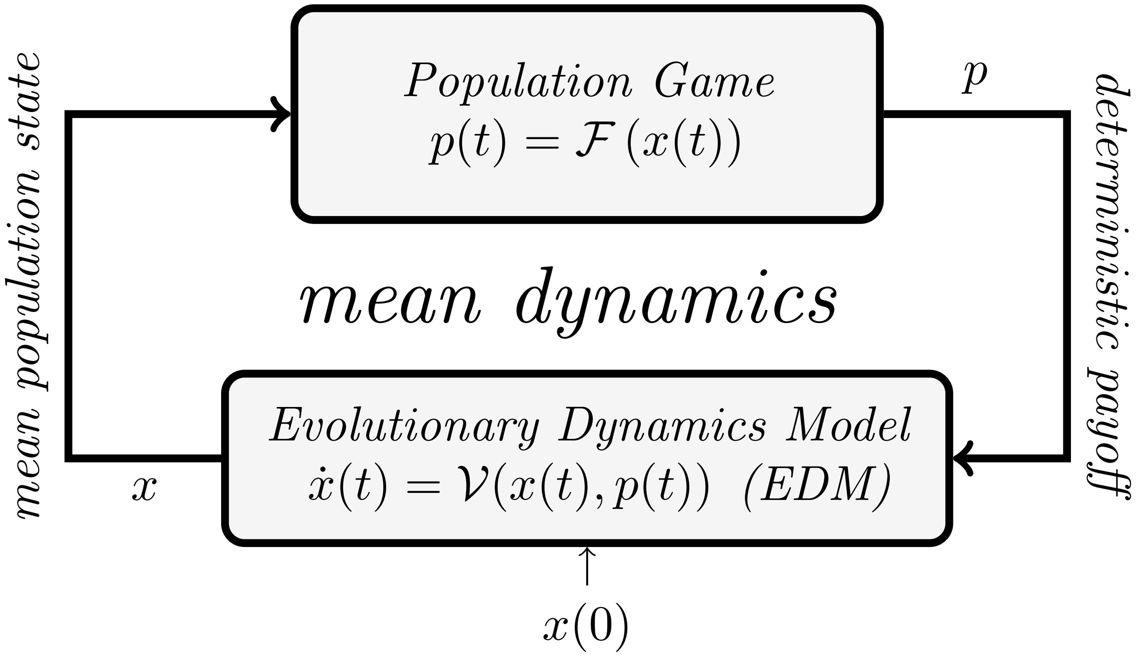
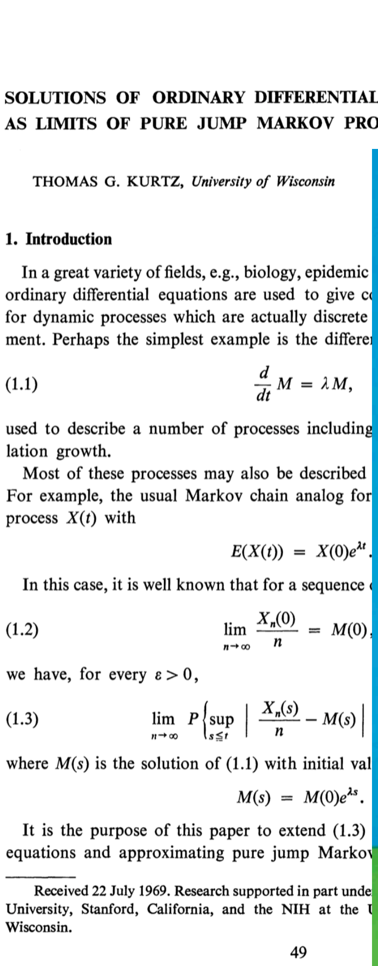


$$\begin{bmatrix} X^N(t) \\ P^N(t) \end{bmatrix} \xrightarrow{N \to \infty} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}$$

Convergence Guarantees


J. Appl. Prob. 7, 49–58 (1970)
Printed in Israel

SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS
AS LIMITS OF PURE JUMP MARKOV PROCESSES

THOMAS G. KURTZ, University of Wisconsin

1. Introduction

In a great variety of fields, e.g., biology, epidemic
ordinary differential equations are used to give c
for dynamic processes which are actually discrete
ment. Perhaps the simplest example is the differe

(1.1) $\frac{d}{dt}M = \lambda M,$

used to describe a number of processes including
lation growth.

Most of these processes may also be described
For example, the usual Markov chain analog for
process $X(t)$ with

$E(X(t)) = X(0)e^{\lambda t}.$

In this case, it is well known that for a sequence

(1.2) $\lim_{n\to\infty} \frac{X_n(0)}{n} = M(0),$

we have, for every $\varepsilon > 0$,

(1.3) $\lim_{n\to\infty} P\left\{\sup_{s\leq t} \left| \frac{X_n(s)}{n} - M(s) \right|\right.$

where $M(s)$ is the solution of (1.1) with initial val

$M(s) = M(0)e^{\lambda s}.$

It is the purpose of this paper to extend (1.3)
equations and approximating pure jump Markov

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University, Stanford, California, and the NIH at the U
Wisconsin.

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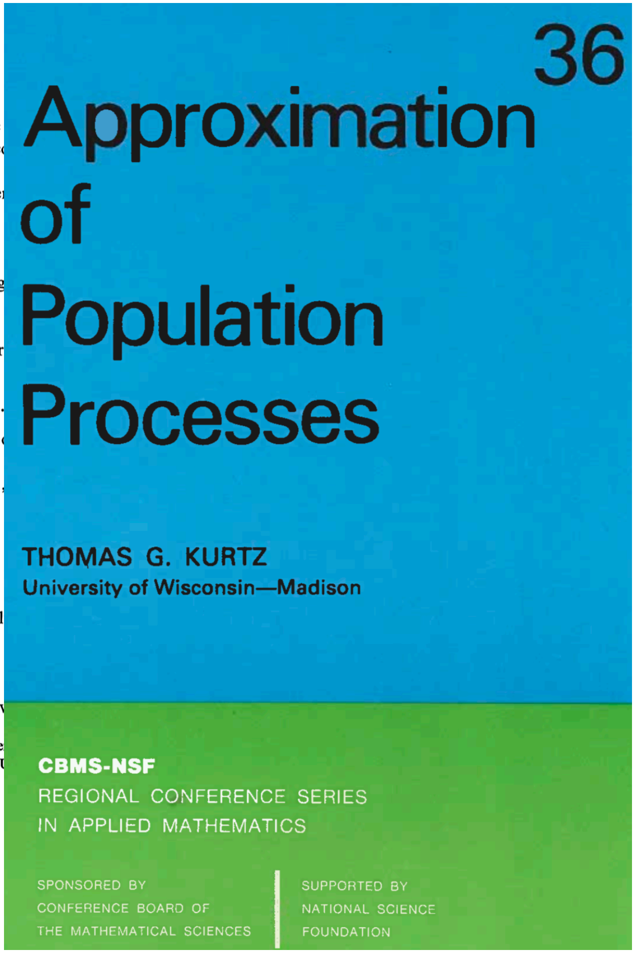
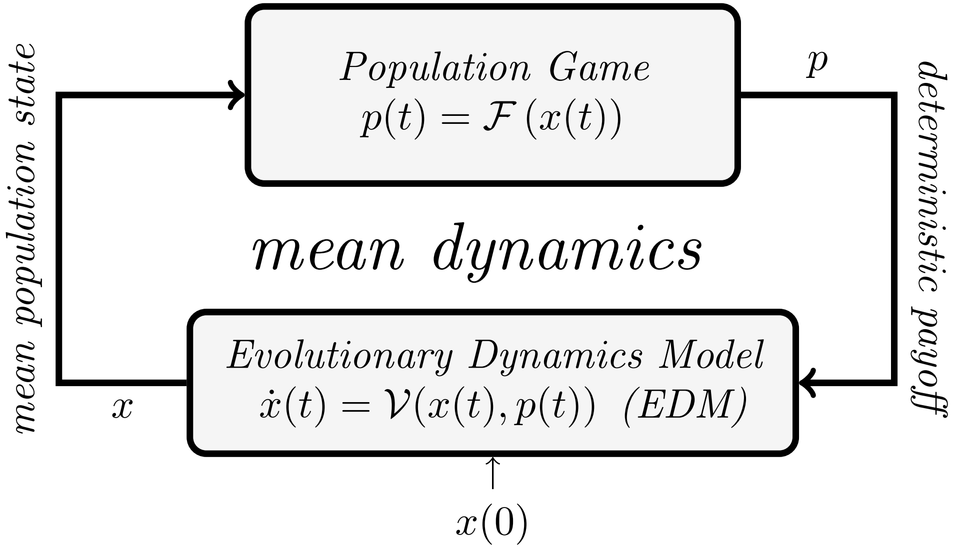


Assume $\lim_{N\to\infty} X^N(0) = x(0)$

For any positive δ and integer T :

$$\lim_{N\to\infty} \Pr\left(\sup_{0\leq t\leq T} \left\| \begin{bmatrix} X^N(t) \\ P^N(t) \end{bmatrix} - \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \right\| > \delta \right) = 0$$



Convergence Guarantees

$$\lim_{N\to\infty} \lim_{t\to\infty} \Pr(X^N(t) \in \mathbb{O}) = 1$$

Ergod. Th. & Dynam. Sys. (1998), **18**, 53–87
Printed in the United Kingdom

Recursive algorithms, urn processes and chaining number of chain recurrent sets

MICHEL BENAÏM

Department of Mathematics, Université Paul Sabati

(*Received 26 January 1996 and accepted in revised*

Abstract. This paper investigates the dynamical properties and recursive stochastic algorithms with constant gain whicl pattern recognition, learning theory, and elsewhere.

It is shown that, under suitable conditions, invariant mea concentrate on the Birkhoff center of irreducible (i.e. chain t vector field $F : \mathbb{R}^d \to \mathbb{R}^d$ obtained by averaging. Appli situations including the cases where F is Axiom A or Mors F is a planar vector field, F has finitely many alpha and or

1. *Introduction*

This paper considers a family of discrete time stochastic proc in $\mathbb{R}^d$ which are defined in the following way.

Let $\mathcal{X} = \{1, \ldots, m\}$ be a finite state space called the *spac* each $x \in \mathbb{R}^d$ we assume that we are given the following:

- a discrete time Markov chain on $\mathcal{X}$ represented by $K(x) = \{K_{i,j}(x)\}_{i,j\in\mathcal{X}}$ satisfying

$$K_{i,j}(x) \geq 0; \quad \sum_{j=1}^{m} K_{i,j}(x) =$$

- a family $\{\mu_x^1, \ldots, \mu_x^m\}$ of m probability measures on T

Let ϵ denote a (small) positive real parameter called the g a Markov process $\{(X_n^\epsilon, \Theta_n^\epsilon)\}_{n\in\mathbb{N}}$ defined on a probability sp in $\mathbb{R}^d \times \mathcal{X}$ whose transition kernel is given by

$$\mathbf{P}((X_{n+1}^\epsilon, \Theta_{n+1}^\epsilon) \in A \times \{j\} \mid X_n^\epsilon = x, \Theta_n^\epsilon = i) = K_{i,}$$

for every $i, j \in \mathcal{X}, x \in \mathbb{R}^d$ and every Borel set $A \subset$ $\{(-x+f)/\epsilon : f \in A\}$.

The Annals of Applied Probability
1999, Vol. 9, No. 1, 216–241

STOCHASTIC APPROXIMATION ALGORITHMS WITH CONSTANT STEP SIZE WHOSE AVERAGE IS COOPERATIVE[1]

BY MICHEL BENAÏM AND MORRIS W. HIRSCH[2]

Université Paul Sabatier and University of California, Berkeley

We consider stochastic approximation algorithms with constant step size whose average ordinary differential equation (ODE) is cooperative and irreducible. We show that, under mild conditions on the noise process, invariant measures and empirical occupations measures of the process weakly converge (as the time goes to infinity and the step size goes to zero) toward measures which are supported by stable equilibria of the ODE. These results are applied to analyzing the long-term behavior of a class of learning processes arising in game theory.

0. Introduction. *Stochastic approximation algorithms with constant step size* are discrete time stochastic processes whose general form can be written as

$$X_{n+1}^\varepsilon - X_n^\varepsilon = \varepsilon f(X_n^\varepsilon, \xi_{n+1}), \tag{1}$$

where X_n^ε lives in $\mathbf{R}^m$, $\{\xi_n\}_{n\in\mathbf{N}}$ is a stochastic process, f is a suitable function and ε a small positive parameter (the step size).

Processes described by (1) appear in a large variety of domains such as system identification or control theory; they encompass several models of learning and adaptive behavior in neural network, game theory and elsewhere.

To analyze the asymptotic behavior of (1) it is often convenient to introduce an ordinary differential equation (ODE)

$$\frac{dx}{dt} = F(x) \tag{2}$$

obtained from (1) by suitable averaging. This method, called the *method of ordinary differential equation*, was introduced by Ljung (1977) and widely studied thereafter [see, e.g., Kushner and Clark (1978), Benveniste, Métivier and Priouret (1990), Duflo (1997)]. Until recently, however, most of the work in this direction has assumed the simplest dynamics for F (for example that F is the negative of the gradient of a cost function), and little attention has been paid to dynamical systems issues.

Recent works by Benaïm (1996a, b), Benaïm and Hirsch (1995, 1996), Duflo (1996) and Fort and Pages (1997) have shown how the long-term behavior of stochastic approximation algorithms can be precisely related to the long-

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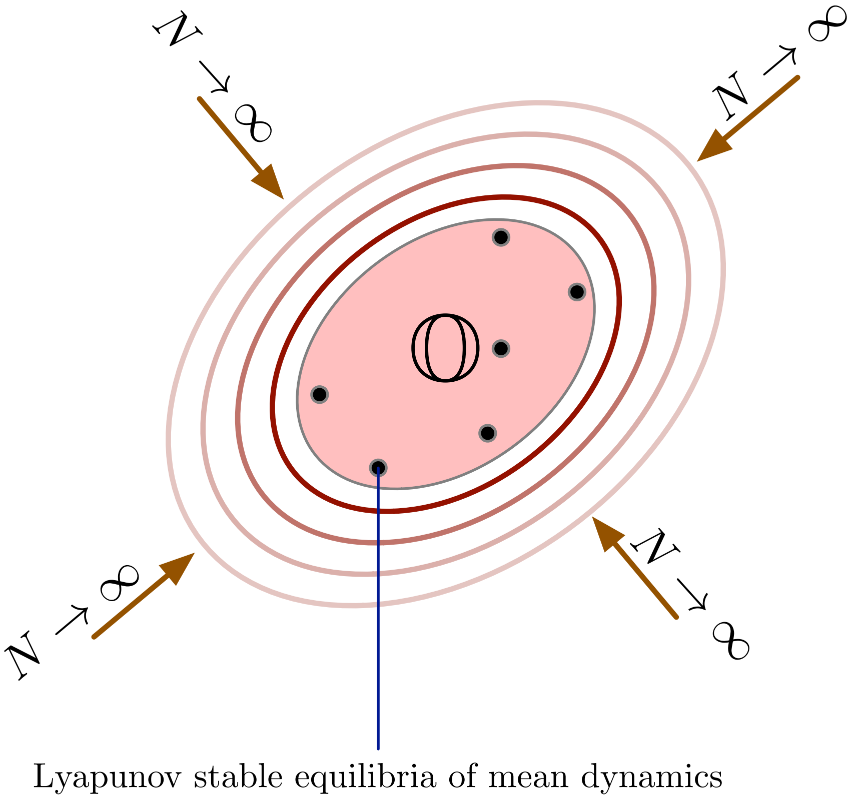
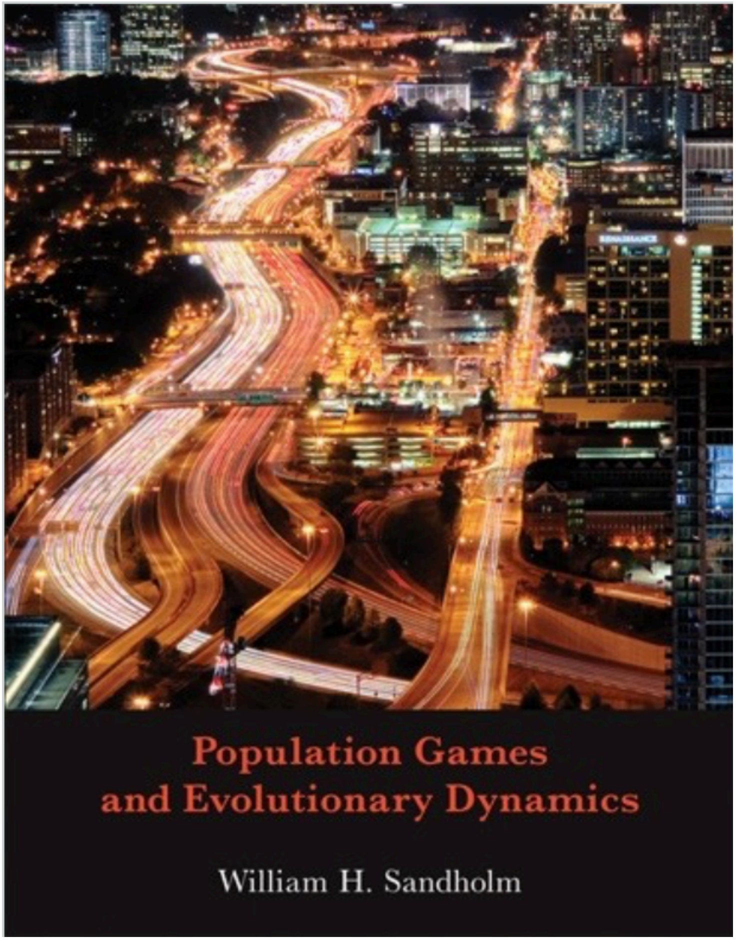
[1]Supported by NATO Grant CRG 95-0857.
[2]Supported in part by an NSF grant.
AMS 1991 *subject classifications*. Primary 62L20; secondary 34C35, 34F05, 93E35.
Key words and phrases. Stochastic approximation, ordinary differential equation method, cooperative vector fields, large deviations, weak convergence, theory of learning in games.

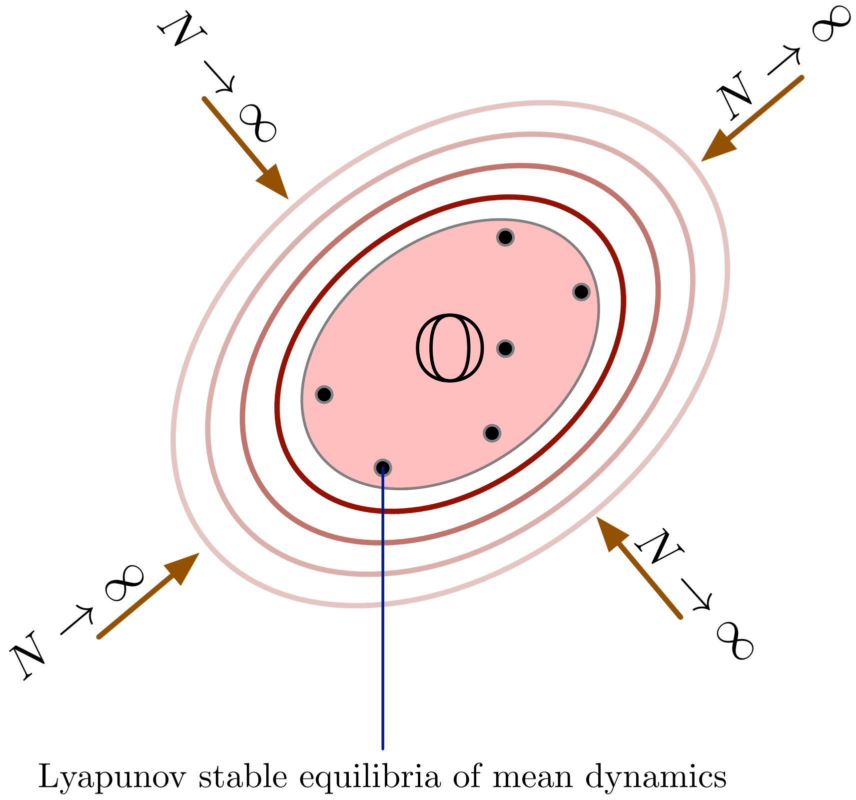


Convergence Guarantees

12.B Appendix: Stochastic Approximation Theory

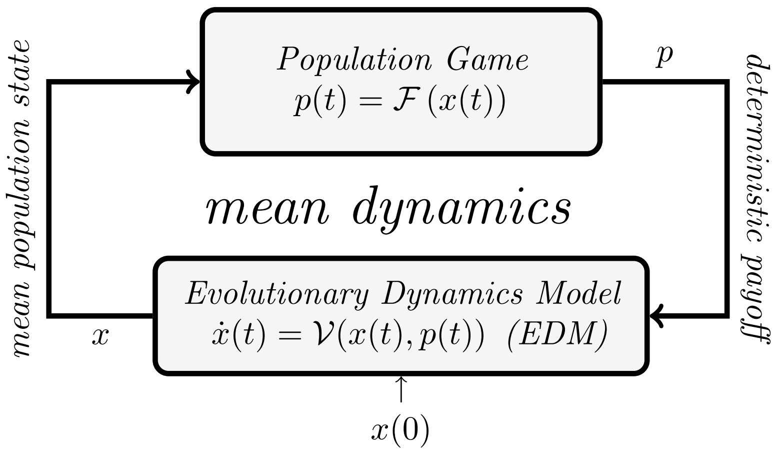


$$\lim_{N\to\infty} \lim_{t\to\infty} \Pr(X^N(t) \in \mathbb{O}) = 1$$



Nash Stationarity

Main goal: characterize stable equilibria of the mean dynamics



rest points or equilibria of the mean dynamics

$$\mathbb{O}^{\mathcal{F},\mathcal{V}} := \left\{ z \in \mathbb{X} \;\middle|\; \mathcal{V}(z, \mathcal{F}(z)) = 0 \right\}$$

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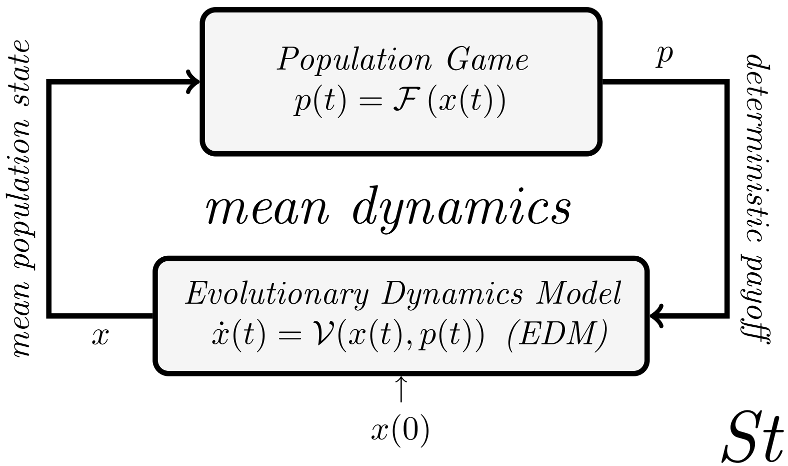

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Stability concepts

(Global Attractiveness) The set $\mathbb{O}^{\mathcal{F},\mathcal{V}}$ is <u>globally attractive</u> if for every initial condition $x(0)$ in $\mathbb{X}$, $x(t)$ satisfies:

$$\lim_{t \to \infty} \inf_{z \in \mathbb{O}^{\mathcal{F},\mathcal{V}}} \|x(t) - z\| = 0$$

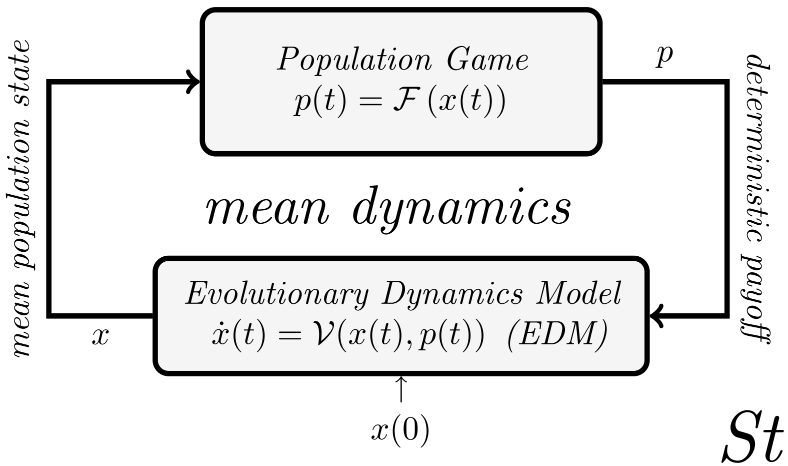
(Lyapunov Stability) The set $\mathbb{O}^{\mathcal{F},\mathcal{V}}$ is Lyapunov stable if for every open set $\mathbb{O}$ in $\mathbb{R}^n$ that contains $\mathbb{O}^{\mathcal{F},\mathcal{V}}$, there is another open set $\mathbb{O}'$ for which the following holds:

$$x(0) \in \mathbb{O}' \cap \Delta \implies x(t) \in \mathbb{O}, \; t \geq 0$$

(Global Asymptotic Stability) The set $\mathbb{O}^{\mathcal{F},\mathcal{V}}$ is globally asymptotically stable if it is globally attractive and Lyapunov stable.

Nash Stationarity

Main goal: characterize stable equilibria of the mean dynamics



rest points or equilibria of the mean dynamics

$$\mathbb{O}^{\mathcal{F},\mathcal{V}} := \left\{ z \in \mathbb{X} \;\middle|\; \mathcal{V}(z, \mathcal{F}(z)) = 0 \right\}$$

Stability concepts

(Global Attractiveness) The set $\mathbb{O}^{\mathcal{F},\mathcal{V}}$ is globally attractive if for every initial condition $x(0)$ in $\mathbb{X}$, $x(t)$ satisfies:

$$\lim_{t \to \infty} \inf_{z \in \mathbb{O}^{\mathcal{F},\mathcal{V}}} \|x(t) - z\| = 0$$

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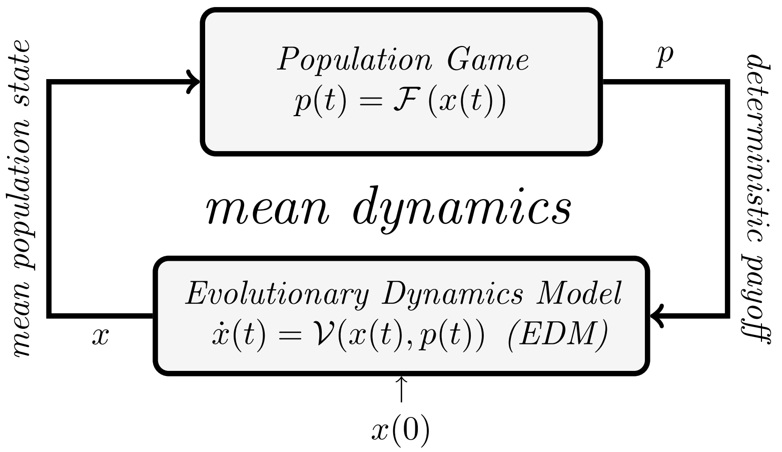

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$$\mathcal{V}(z, r) = 0 \Leftrightarrow z \in \arg\max_{\bar{z} \in \mathbb{X}} \bar{z}^T r, \quad z \in \mathbb{X},\ r \in \mathbb{R}^n$$

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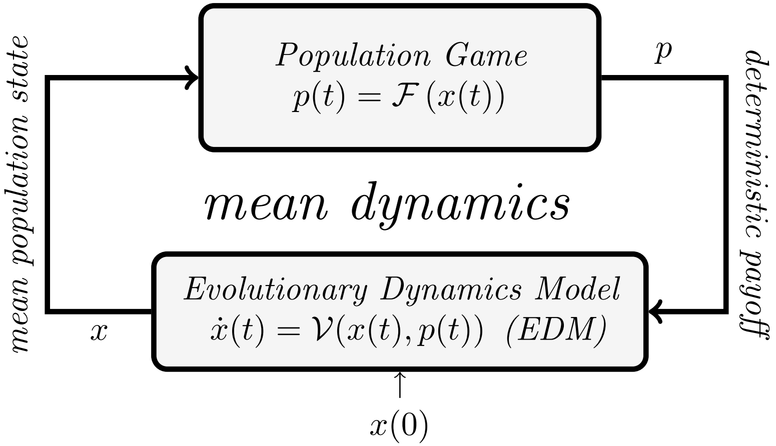

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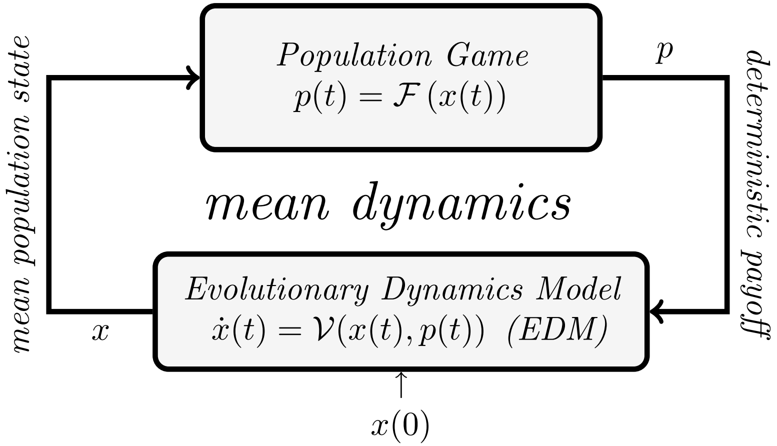

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Example: Smith EDM satisfies Nash stationarity

$$\mathcal{T}_{ij}^{Smith}(z, r) := [r_j - r_i]_+, \quad r \in \mathbb{R}^n, \quad z \in \mathbb{X}$$

$$\mathcal{V}_i^{Smith}(z, r) := \sum_{j=1}^{n} z_j [r_i - r_j]_+ - \left(\sum_{j=1}^{n} [r_j - r_i]_+ \right) z_i$$

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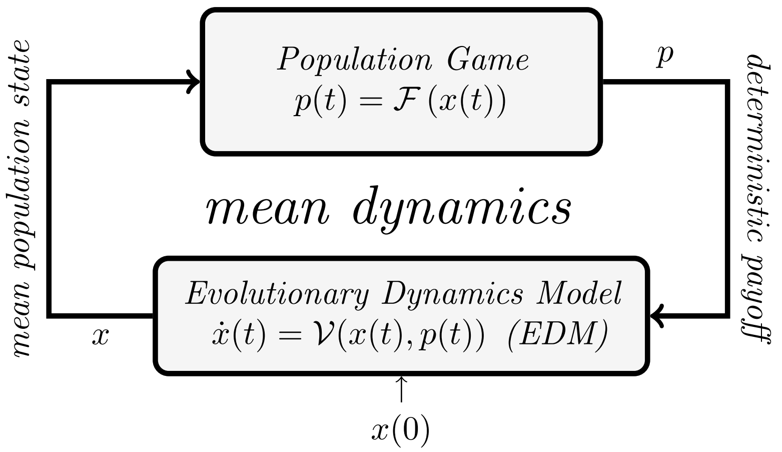
Main goal: characterize stable equilibria of the mean dynamics



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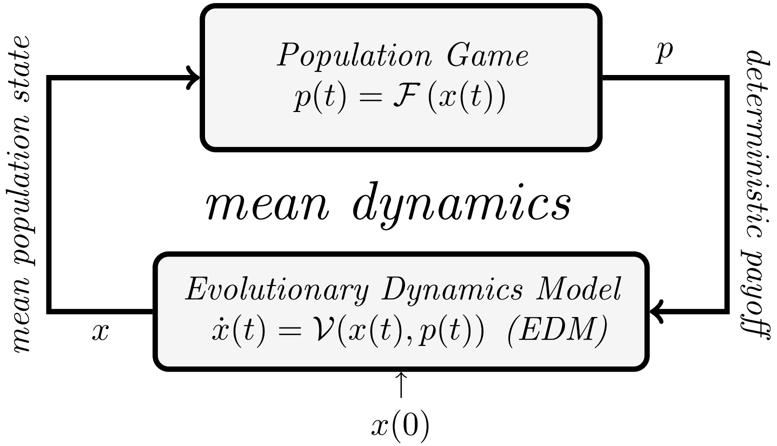
Example: The Brown-von Neumann-Nash (BNN) EDM specified by the following separable EPT protocol also satisfies NS:

$$\mathcal{T}_{ij}^{\mathrm{BNN}}(\hat{r}) := [\hat{r}_j]_+, \quad \hat{r}_i := r_i - \sum_{j=1}^{n} r_i z_i$$

excess payoff

Nash Stationarity

Main goal: characterize stable equilibria of the mean dynamics



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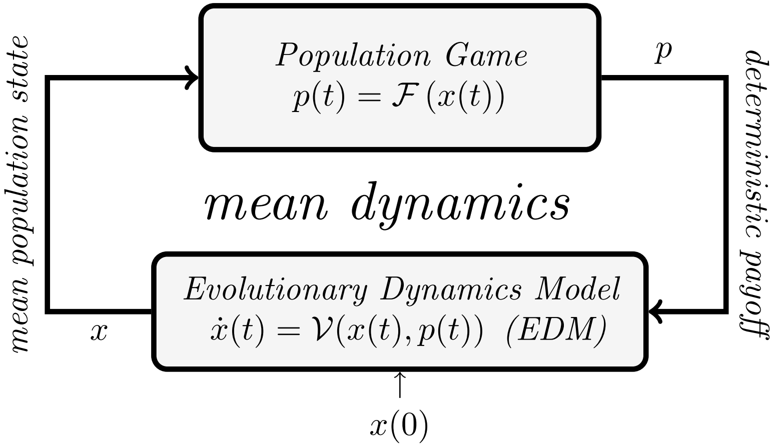

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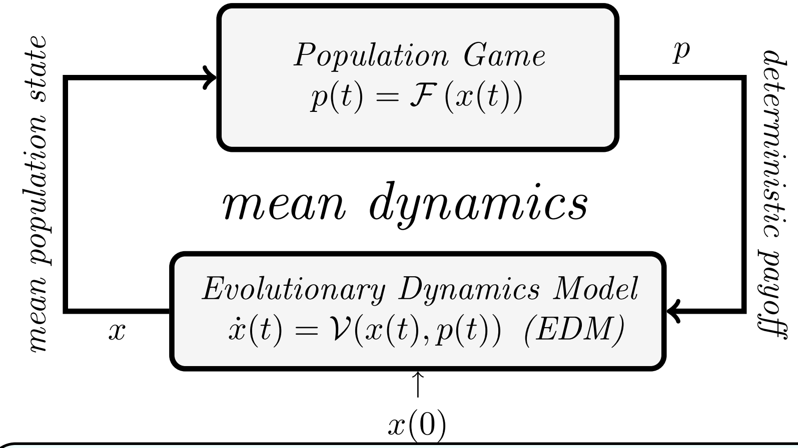
Positive Correlation

Main goal: characterize stable equilibria of the mean dynamics



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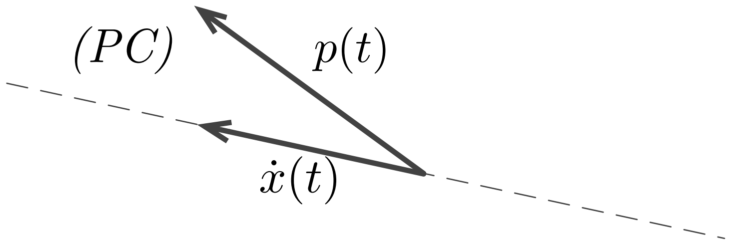
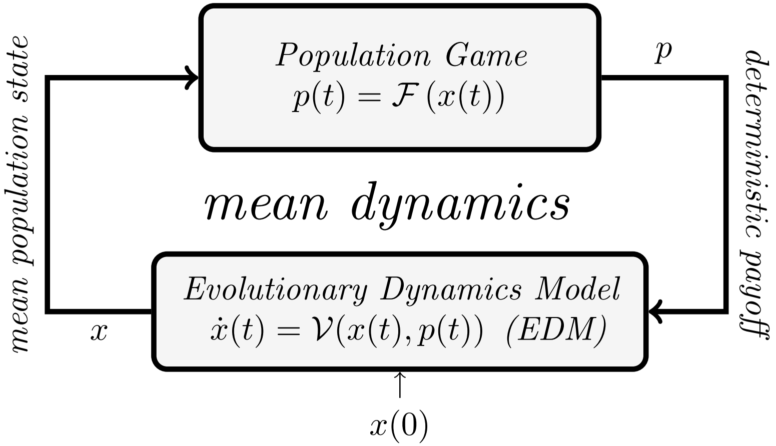
$$\Big(\mathcal{V}(z, r) \neq 0 \text{ and } r \neq 0\Big) \Rightarrow r^T \mathcal{V}(z, r) > 0$$



Positive Correlation

Main goal: characterize stable equilibria of the mean dynamics



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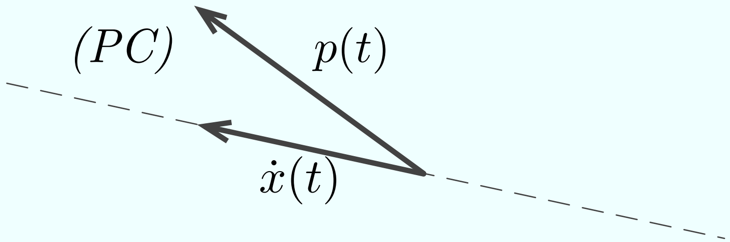
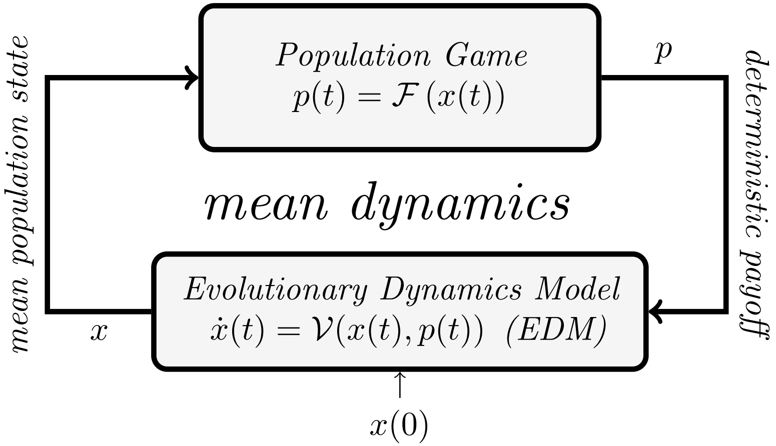
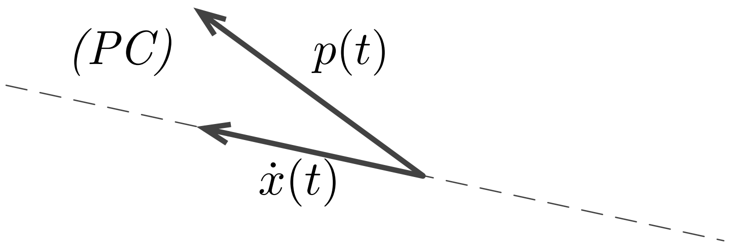
Smith and BNN EDM satisfy PC



Contractive Population Games

Main goal: characterize stable equilibria of the mean dynamics





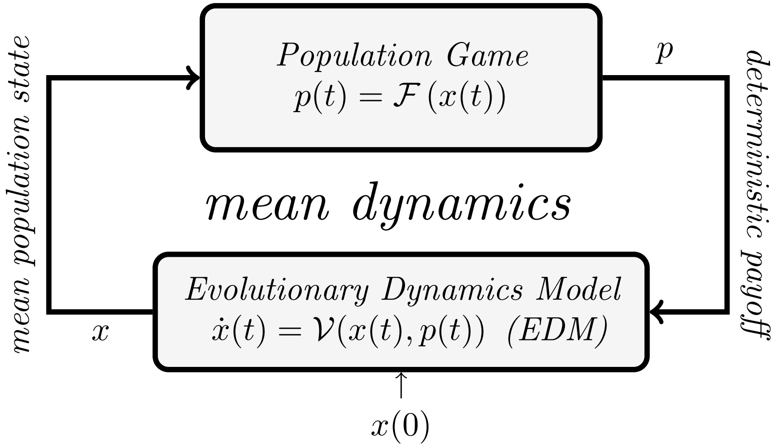
(Contractive Population Game) *A given population game $\mathcal{F} : \mathbb{X} \to \mathbb{R}^n$ is qualified as contractive if it satisfies the following condition:*

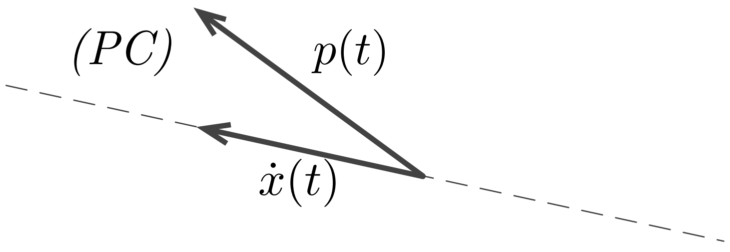
$$(z - \bar{z})^T \big(\mathcal{F}(z) - \mathcal{F}(\bar{z})\big) \leq 0, \quad z, \bar{z} \in \mathbb{X}$$

The population game is said to be strictly contractive if the inequality above is strict for $z \neq \bar{z}$.

Contractive Population Games

Main goal: characterize stable equilibria of the mean dynamics





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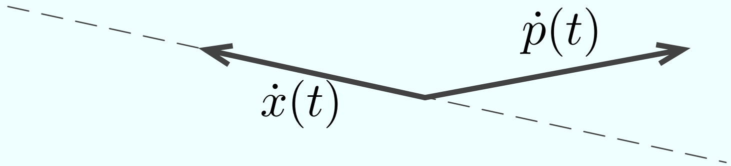
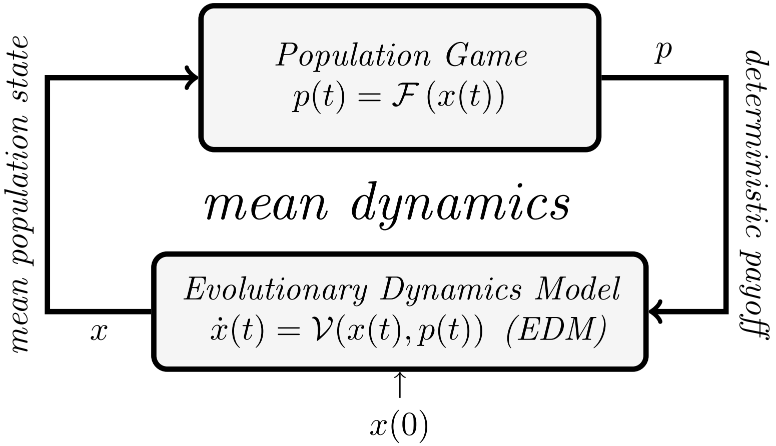
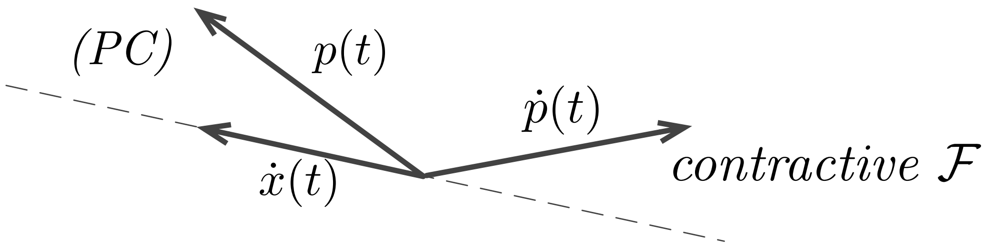


$$\dot{p}(t)^T \dot{x}(t) \leq 0$$

Contractive Population Games

Main goal: characterize stable equilibria of the mean dynamics





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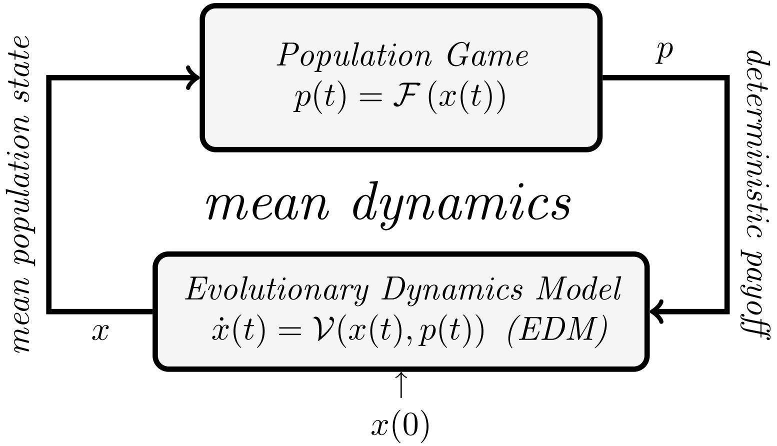
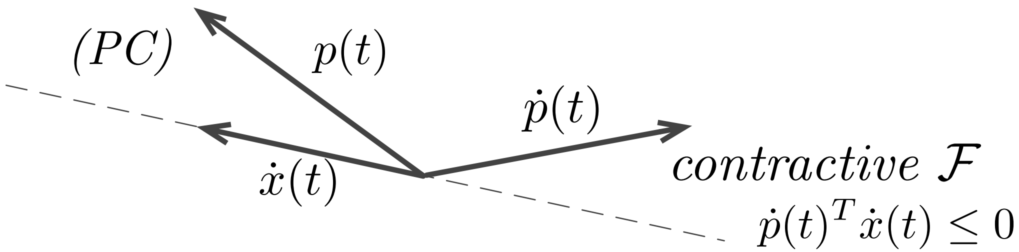
If $\mathcal{F}$ is contractive (strictly contractive) with a unique Nash equilibrium $\mathbb{NE}(\mathcal{F}) = \{x^*\}$ then x^* is a GNESS (GESS).

Contractive Population Games

Main goal: characterize stable equilibria of the mean dynamics



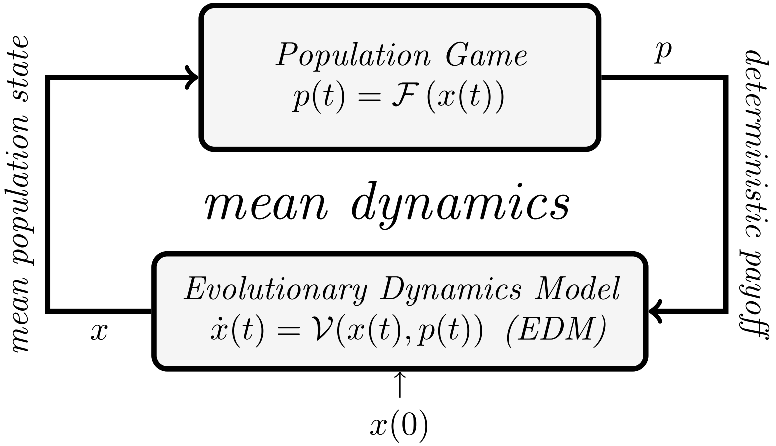


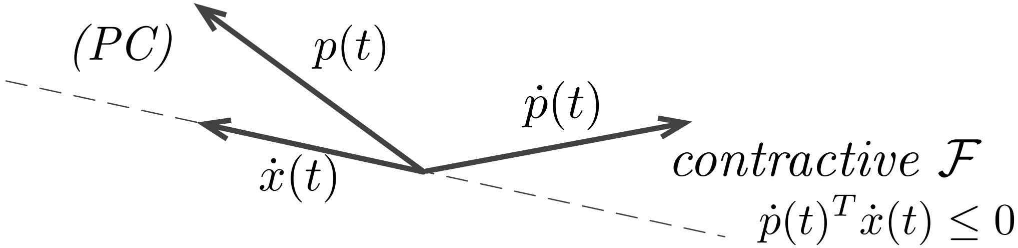
Examples of contractive games

- *Potential concave (strictly concave) games.*
- *Random matching of symmetric zero-sum games (null stable).*
- *Wars of attrition.*

Contractive Population Games

Main goal: characterize stable equilibria of the mean dynamics





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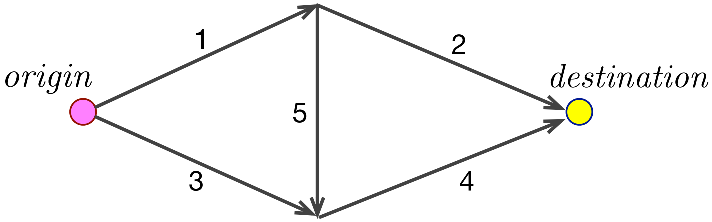

$$\mathcal{F}_1(z) = -\mathcal{D}_1(z_1 + z_2) - \mathcal{D}_2(z_1)$$
$$\mathcal{F}_2(z) = -\mathcal{D}_1(z_1 + z_2) - \mathcal{D}_4(z_2 + z_3) - \mathcal{D}_5(z_2)$$
$$\mathcal{F}_3(z) = -\mathcal{D}_3(z_3) - \mathcal{D}_4(z_2 + z_3)$$

$$D\mathcal{F}(z) = \begin{bmatrix} -\dot{\mathcal{D}}_1(z_1+z_2) - \dot{\mathcal{D}}_2(z_1) & -\dot{\mathcal{D}}_1(z_1+z_2) & 0 \\ -\dot{\mathcal{D}}_1(z_1+z_2) & -\dot{\mathcal{D}}_1(z_1+z_2) - \dot{\mathcal{D}}_4(z_2+z_3) - \dot{\mathcal{D}}_5(z_2) & -\dot{\mathcal{D}}_4(z_2+z_3) \\ 0 & -\dot{\mathcal{D}}_4(z_2+z_3) & -\dot{\mathcal{D}}_3(z_3) - \dot{\mathcal{D}}_4(z_2+z_3) \end{bmatrix}$$

Contractive P.G.: Stability Under BNN

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Stable games and their dynamics

Josef Hofbauer [a], William H. Sandholm [b,*]

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Abstract

We study a class of population games called *stable games*. These games are characterized by *self-defeating externalities*: when agents revise their strategies, the improvements in the payoffs of strategies to which revising agents are switching are always exceeded by the improvements in the payoffs of strategies which revising agents are abandoning. We prove that the set of Nash equilibria of a stable game is globally asymptotically stable under a wide range of evolutionary dynamics. Convergence results for stable games are not as general as those for potential games: in addition to monotonicity of the dynamics, integrability of the agents' revision protocols plays a key role.

JEL classification: C72; C73

Keywords: Population games; Evolutionarily stable strategies; Evolutionary dynamics; Global stability; Lyapunov functions

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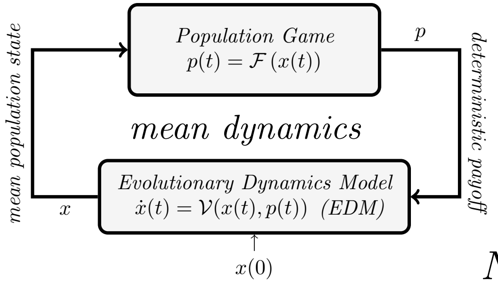
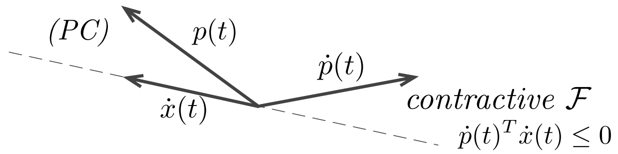
doi:10.1016/j.jet.2009.01.007





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Let us investigate stability for the BNN dynamics assuming that $\mathcal{F}$ is C^1 and contractive

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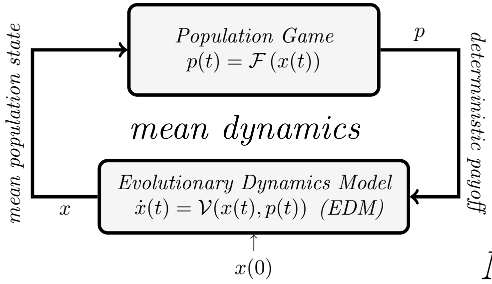

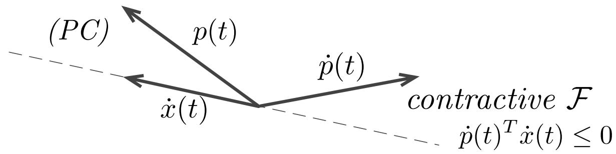

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$$\mathcal{V}_i^{BNN}(z, r) := [\hat{r}_i]_+ - z_i \sum_{j=1}^{n} [\hat{r}_j]_+$$

Revision potential of the BNN protocol:

$$\mathcal{I}^{BNN}(\hat{r}) = \frac{1}{2} \sum_{i=1}^{n} [\hat{r}_i]_+^2, \quad z \in \mathbb{X},\ r \in \mathbb{R}^n$$

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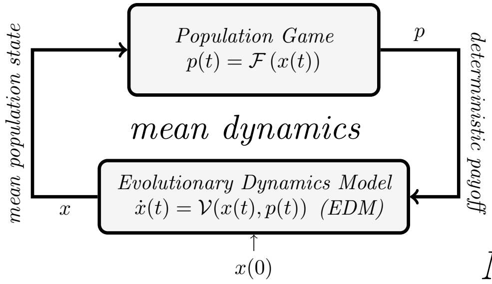

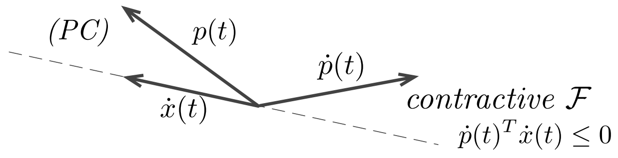

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Contractive P.G.: Stability Under BNN

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Stable games and their dynamics

Josef Hofbauer[a], William H. Sandholm[b,*]

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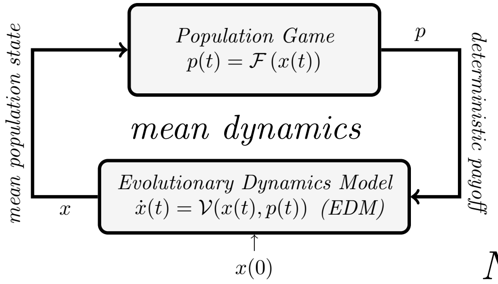

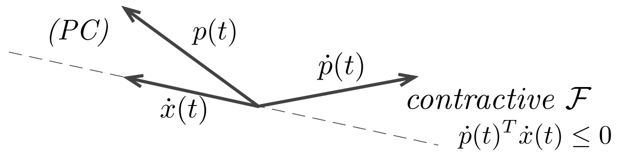

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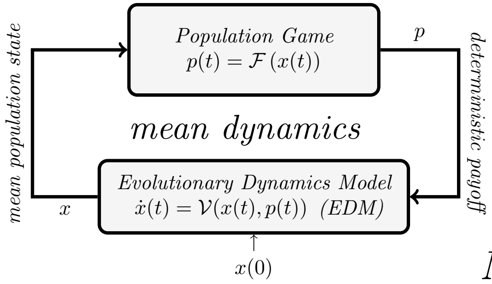

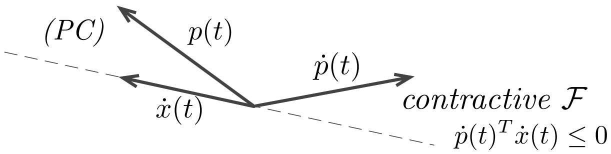

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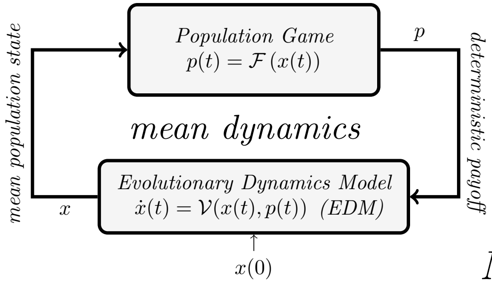

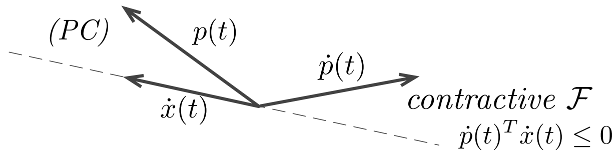

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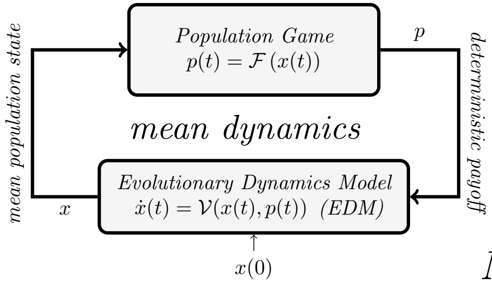

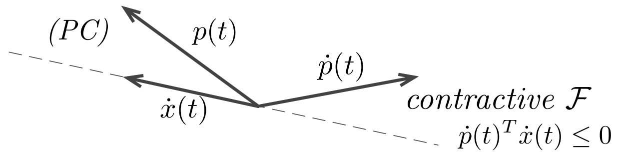

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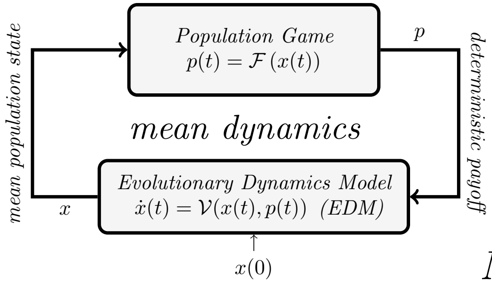

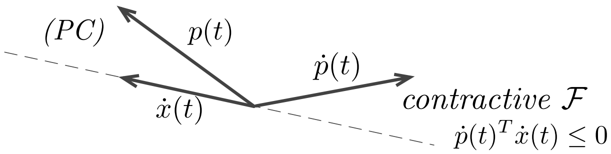

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Contractive P.G.: Stability Under EPT

(Excess Payoff Target (EPT) EDM)
A given protocol $\mathcal{T}$ yields an EPT EDM if it can be written as:

$$\left.\begin{aligned} \mathcal{T}_{ij}(r,z) &= \mathcal{T}_j^{EPT}(\hat{r}) \\ \hat{r}_i &\stackrel{\text{def}}{=} r_i - \tfrac{1}{m}\textstyle\sum_{i=1}^n r_i z_i \end{aligned}\right. , \quad r \in \mathbb{R}^n, \; z \in \mathbb{X}$$

where $\mathcal{T}^{EPT} : \mathbb{R}^n_* \to \mathbb{R}^n_+$ is a Lipschitz continuous map, $\hat{r}$ is the vector of excess payoff relative to the population average and $\mathbb{R}^n_* = \mathbb{R}^n - \mathrm{int}(\mathbb{R}^n_-)$ is the set of possible excess payoff vectors. In addition, $\mathcal{T}^{EPT}$ must satisfy the following acuteness condition:

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(Integrable EPT EDM) *A given EPT protocol $\mathcal{T}^{EPT} : \mathbb{R}^n_* \to \mathbb{R}^n_+$ is integrable if there is a continuously differentiable function $\mathcal{I}^{EPT} : \mathbb{R}^n \to \mathbb{R}$ such that the following holds:*

$$\mathcal{T}^{EPT}(\hat{r}) = \nabla \mathcal{I}^{EPT}(\hat{r}), \quad \hat{r} \in \mathbb{R}^n_*$$

We refer to $\mathcal{I}^{EPT}$ as the revision potential of $\mathcal{T}^{EPT}$.

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Integrable EPT EDM satisfies NS and PC.

Contractive P.G.: Stability Under EPT

Global asymptotic stability of $\mathbb{NE}$ for EPT EDM

Theorem *(Park, Martins, Shamma in CDC'19 and ArXiv'19)*

Consider that $\mathcal{F}$ is C^1 and contractive. Let a mean dynamics be formed by an integrable EPT EDM and $\mathcal{F}$. The set $\mathbb{NE}(\mathcal{F})$ is globally asymptotically stable.

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Contractive P.G.: Stability Under IPC

(Impartial Pairwise Comparison (IPC) EDM) *A given protocol $\mathcal{T}$ yields an impartial pairwise comparison (IPC) EDM if it can be written as:*

$$\mathcal{T}_{ij}(r,z) = \mathcal{T}^{IPC}_j(r_j - r_i), \quad r \in \mathbb{R}^n,\ z \in \mathbb{X}$$

where $\mathcal{T}^{IPC} : \mathbb{R}^n \to \mathbb{R}^n_+$ is a Lipschitz continuous map, which also satisfies the following sign preservation condition:

$$\begin{cases} \mathcal{T}^{IPC}_j(r_j - r_i) > 0, & \textit{if} \ \ r_j > r_i \\ \mathcal{T}^{IPC}_j(r_j - r_i) = 0, & \textit{if} \ \ r_j \le r_i \end{cases}, \quad r \in \mathbb{R}^n$$

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where $\mathcal{T}^{IPC} : \mathbb{R}^n \to \mathbb{R}^n_+$ is a Lipschitz continuous map, which also satisfies the following sign preservation condition:

$$\begin{cases} \mathcal{T}_j^{IPC}(r_j - r_i) > 0, & \text{if} \ \ r_j > r_i \\ \mathcal{T}_j^{IPC}(r_j - r_i) = 0, & \text{if} \ \ r_j \le r_i \end{cases}, \quad r \in \mathbb{R}^n$$

IPC EDM satisfies NS and PC.

Contractive P.G.: Stability Under IPC

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The following is a Lyapunov function

$$\mathcal{L}(z) := \sum_{i=1}^{n} \sum_{j=1}^{n} z_i \int_0^{\mathcal{F}_j(z) - \mathcal{F}_i(z)} \mathcal{T}_j^{\mathrm{IPC}}(\tau)\ \mathrm{d}\tau, \quad z \in \mathbb{X}$$

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Example:

$$\mathcal{T}_{ij}^{Smith}(z,r) := [r_j - r_i]_+, \quad r \in \mathbb{R}^n, \quad z \in \mathbb{X}$$

$$\mathcal{L}(z) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} z_i [\mathcal{F}_j(z) - \mathcal{F}_i(z)]_+^2$$

Contractive P.G.: Stability Under IPC

Theorem *(Hofbauer and Sandholm, 2009)*

Consider that $\mathcal{F}$ is C^1 and contractive. Let a mean dynamics be formed by an integrable IPC EDM and $\mathcal{F}$. The set $\mathbb{NE}(\mathcal{F})$ is globally asymptotically stable.

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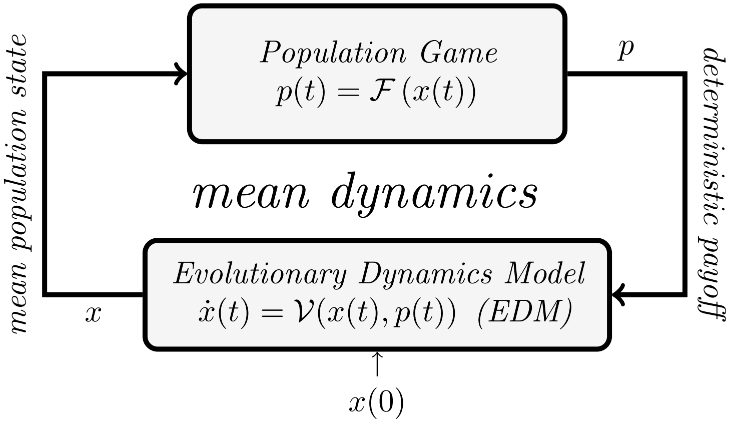
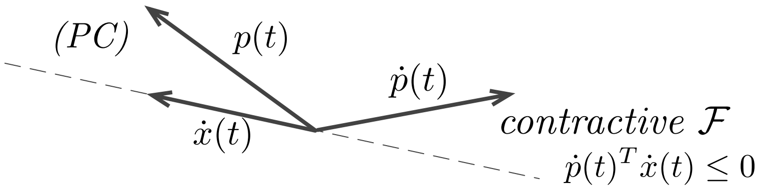
Brief Summary and Thanks

Recap of basic formulation



EDM Concepts: NS and PC

Contractive games



Systematic method to establish GAS of $\mathbb{NE}(\mathcal{F})$

Thanks for support by

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