Payoff dynamics and higher-order learning in population games

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Evolutionary Game Theory

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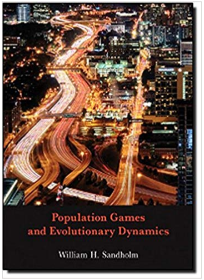
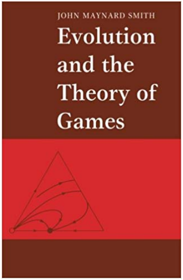
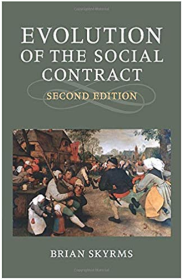
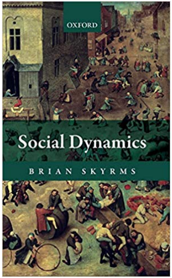
Population Games + Evolutionary Dynamics

Modeling strategic interactions among large numbers









Networks, Evolutionary Biology, Social Learning,...

- Distribution of subpopulation types:

$$x = (x_1, x_2, ..., x_n) \in \Delta$$

x_i is fraction of type i

Population games: Setup

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$$\dot{x}_i = \underbrace{\left(F_i(x) - \overbrace{x^{\mathrm{T}} F(x)}^{\substack{\text{average} \\ \text{fitness}}}\right)}_{\text{growth rate}} x_i$$

Evolutionary dynamics

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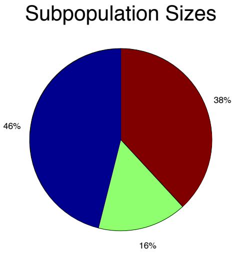
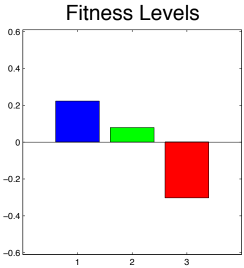
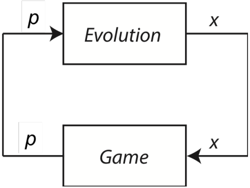
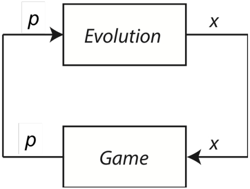






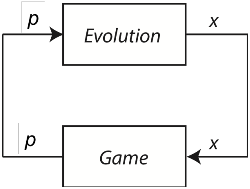
- *Viewpoint:* Detach population game from dynamics
  - Evolution: Maps payoffs to distributions
  - Game: Maps distributions to payoffs
- *Replicator revisited:*

$$\dot{x}_i = (p_i - x^{\mathrm{T}} p) x_i$$
$$p = F(x)$$



Appeal:

- Bring to bear arsenal of feedback system analysis tools
- Address broader framework beyond traditional scope



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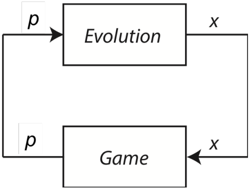
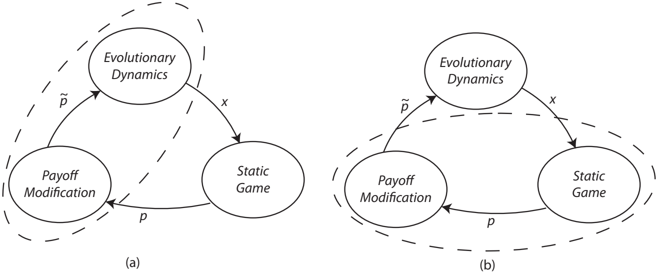
Specific interest: "Higher order" Evolutionary Game Theory


p
Evolution
x
p
Game
x

vs


Evolutionary Dynamics
$\tilde{p}$
x
Payoff Modification
Static Game
p
(a)
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(b)

Population games: Motivation and foundational concepts
Shamma

Stability analysis: Potential and contractive games
Martins

From population games to payoff dynamics models: A passivity-based approach
Park

- Finite number of players: N
- Finite set of player strategies (was "types"): $S = \{1, 2, ..., n\}$
- Fraction of users using strategy i: x_i
- Population distribution:

$$x = (x_1, x_2, ..., x_n) \in \Delta$$

Finite population games

- Finite number of players: N
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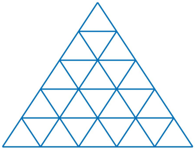
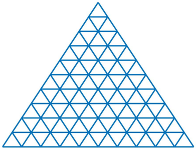
- Finiteness implies discretized subset of Δ



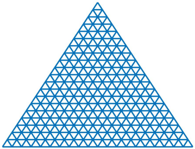
$N = 5$

$n = 3$



$N = 10$

$n = 3$



$N = 20$

$n = 3$

- Vector of fitness levels:

$$p = (p_1, p_2, ..., p_n)$$

- Fitness is function of population:

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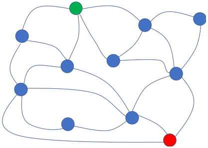
$$p = F(x)$$

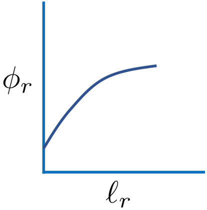
As before, $F(\cdot)$ evaluated on discretized subset of Δ

- *Feature:* Anonymity among agents

- Strategy = Path from start to finish, e.g.,

$$i = \{r_1, r_2, ..., r_m\}$$





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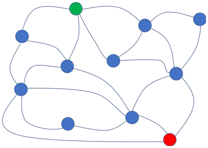
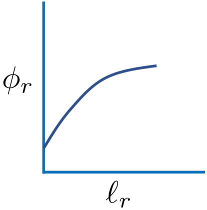
- Load of road ℓ_r depends on paths that use r:

$$\ell_r(x) = \sum_{i:r\in i} x_i$$





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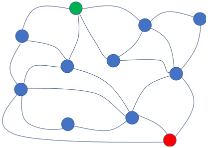

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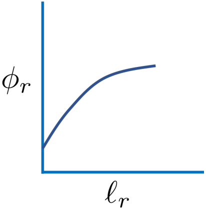
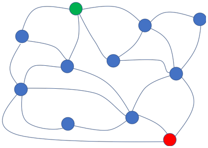
- Congestion on a road depends on overall load:

$$\phi_r(\ell_r(x))$$





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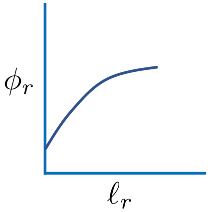

- Fitness of strategy (path) i:

$$F_i(x) = -\sum_{r\in i} \phi_r(\ell_r(x))$$

Note: Negative sign turns penalty into reward.

- A "language" $\mathcal{L}$ is a pair of matrices (P, Q)
  - Binary elements, row sum $= 1$
  - Speaker matrix: P : events $\rightarrow$ words
  - Hearer matrix: Q : words $\rightarrow$ events
- Illustration:

$$P = \begin{array}{cccc} & \alpha & \beta & \gamma \\ A & 1 & 0 & 0 \\ B & 1 & 0 & 0 \\ C & 0 & 1 & 0 \end{array} \qquad Q = \begin{array}{cccc} & A & B & C \\ \alpha & 1 & 0 & 0 \\ \beta & 0 & 1 & 0 \\ \gamma & 0 & 0 & 1 \end{array}$$

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- Fitness of language $\mathcal{L}_i = (P_i, Q_i)$:

$$\begin{aligned} F_i(x) &= \mathrm{tr}\left[\frac{1}{N}\sum_{m=1}^{N}(P_i Q_m + P_m Q_i)\right] \quad \text{(agent level)} \\ &= \mathrm{tr}\left[\sum_{k=1}^{n} x_k (P_i Q_k + P_k Q_i)\right] \quad \text{(strategy level)} \end{aligned}$$

- Payoff matrix, A:

$$A_{ij} = \text{payoff to } i \text{ vs opponent strategy } j$$

Example: Random pairwise competition

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- Rock-paper-scissors:

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- Rock-paper-scissors:

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- Fitness is expected payoff to random opponent:

$$F_i(x) = \sum_{j=1}^{n} A_{ij} x_j$$

or

$$F(x) = Ax$$

- Nash equilibrium, $x^* \in \Delta$:
  - Let $p^* = F(x^*)$
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- Examples:
  - Congestion: All active paths have equal congestion.
  - Language:
    - Unanimous use of self-consistent language, i.e., $P = Q^{\mathrm{T}}$
    - Interior NE possible
  - RPS: $(1/3, 1/3, 1/3)$

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- *Feature:* Myopic & memoryless

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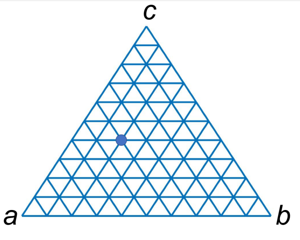
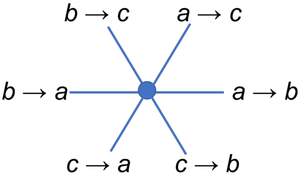
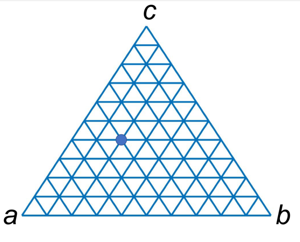
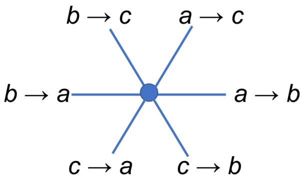




- Revision protocol induces a Markov Chain over discretized simplex





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- Analysis questions:
  - Asymptotic behavior?
  - Transient behavior?
  - For large N?

- Associate discrete stages with continuous time:

$$\tau = 0, 1, 2, \ldots \iff t = 0, \frac{1}{N}, \frac{2}{N}, \ldots$$

Note: Dependence on N

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  Note: Dependence on N

- Define number of i-types: $\xi_i = x_i \cdot N$

- Over a continuous time interval, h:
  - Number of events: $N \cdot h$
  - Expected times i^{th}-strategy chosen: $x_i \cdot Nh = \xi_i h$
  - Expected number of switches to j: $\rho_{ij}(x; F) \cdot \xi_i h$

- Expected change over h:

$$
\begin{aligned}
Nx_i(t+h) &= \xi_i(t+h) \\
&= Nx_i(t) + \sum_{j=1}^{n} \underbrace{\rho_{ji}(x)\xi_j h}_{\text{expected from } j} - \sum_{j=1}^{n} \underbrace{\rho_{ij}(x)\xi_i h}_{\text{expected to } j}
\end{aligned}
$$

or (divide by N)

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x_i(t+h) = x_i(t) + h\left(\sum_{j=1}^{n} \rho_{ji}(x;F)x_j - \sum_{j=1}^{n} \rho_{ij}(x;F)x_i\right)
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$$\Downarrow$$

$$\dot{x}_i = \sum_{j=1}^{n} \rho_{ji}(x;F)x_j - \sum_{j=1}^{n} \rho_{ij}(x;F)x_i$$

- *Pairwise proportional difference (Replicator):* $\rho_{ij}(x;F) = x_j[p_j - p_i]_+$

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$$\dot{x} = \sigma_{\mathrm{softMax}}(p; T) - x$$

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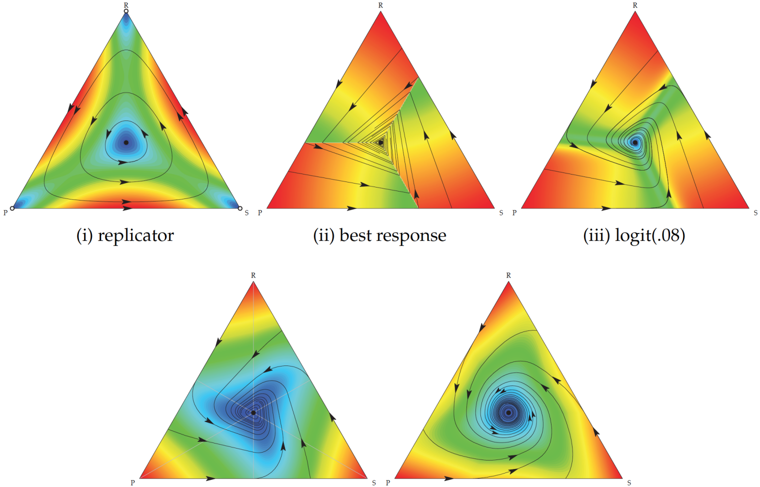
- *Pairwise difference (Smith):* $\rho_{ij}(x;F) = [p_j - p_i]_+$

$$\dot{x}_i = \sum_{j=1}^{n} x_j(p_i - p_j)_+ - x_i \sum_{j=1}^{n} (p_j - p_i)_+$$



(i) replicator (ii) best response (iii) logit(.08)

(iv) BNN (v) Smith

The basic dynamics in standard RPS. Red is fastest, blue is slowest.

- *Setup:*
  - X_t^N: Interpolated[1] solution to stochastic discrete dynamics
  - X_0^N: Initial condition of stochastic dynamics
  - x_t: Solution of mean dynamic
  - x_o: Initial condition of mean dynamic
  - Assume: $\lim_{N\to\infty} X_0^N \to x_o$

[1] $t \in [\tau/N, (\tau+1)/N]$

- *Setup:*
  - X_t^N: Interpolated[1] solution to stochastic discrete dynamics
  - X_0^N: Initial condition of stochastic dynamics
  - x_t: Solution of mean dynamic
  - x_o: Initial condition of mean dynamic
  - Assume: $\lim_{N\to\infty} X_0^N \to x_o$

Theorem

For any $T < \infty$ (horizon) and $\epsilon > 0$ (accuracy),

$$\lim_{N\to\infty} \mathbf{Pr}\left[\sup_{t\in[0,T]} \left|X_t^N - x_t\right| < \epsilon\right] = 1.$$

Benaïm & Weibull, "Deterministic approximation of stochastic evolution in games", *Econometrica*, 2003.

[1] $t \in [\tau/N, (\tau+1)/N]$

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- For large N, mean dynamic approximates stochastic evolution with high probability.
- Deterministic attractor $\Rightarrow$ *Metastable* stochastic state

- *Setup:* Susceptible-Infected-Susceptible (SIS) dynamics
  - Two populations: Susceptible (x) & Infected (1 - x)
  - Revision protocol:

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$\alpha > \beta \implies x^* = \beta/\alpha$ is a global attractor

Caution: Finite horizon vs asymptotic

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- *Comparisons:*
  - Asymptotic behavior: All cured
  - Implication: Arbitrarily long—*but finite*—time at $x^* = \beta/\alpha$

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- *Interpretation:* x fares better than y in the new mix.

$$\dot{x}_i = \underbrace{\Big(F_i(x) - \overbrace{x^{\mathrm{T}} F(x)}^{\substack{\text{average}\\ \text{fitness}}}\Big)}_{\text{growth rate}} x_i$$

Theorem

Let x^* be an ESS. Then x^* is locally asymptotically stable under replicator dynamics.

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- *Upcoming*: **Families** of games + **Families** of dynamics

Sandholm, *Population Games*, 2010.

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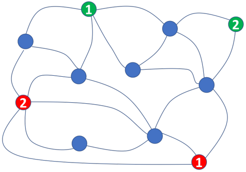

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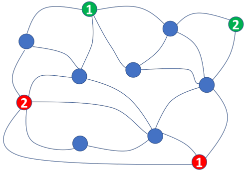

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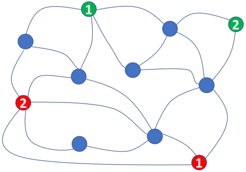

Multiple source-destination pairs.

- *Evolutionary dynamics:*
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  - Deterministic infinite population
- *Analysis:*
  - Asymptotic: Stationary distributions & stochastic stability
  - Finite-horizon: Mean dynamics

Population games: Motivation and foundational concepts
Shamma

Stability analysis: Potential and contractive games
Martins

From population games to payoff dynamics models:
A passivity-based approach
Park